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ON THE TURBULENT HEAT TRANSFER BY FREE CONVECTION FROM A VERTICAL PLATE

HIROHARU KATO, NIICHI NISHIWAKI and MASARU HIRATA

Faculty of Engineering, University of Tokyo, Hongo, Bunkyo-ku, Tokyo, Japan

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Abstract—Until now, the problem on the turbulent heat transfer by free convection has been solved by the profile method whose velocity and temperature profiles are given by experiments. In this paper, shear stress, heat flux, and eddy diffusivity distribution are assumed using the result of forced convection heat transfer, then the coupled partial differential equations for velocity and temperature which are reduced to simpler forms, can be easily solved by the trial and error method. The result can be applied for a wider range of Grashof and Prandtl numbers than that obtained from usual equations.

On transition from laminar to turbulent, a certain dimensionless criterion is proposed, which gives the ratio of energy of mean flow and that dispersed near the wall. The same dimensionless value can be applied for both forced and free convection.

NOMENCLATURE

a, thermal diffusivity [m²/h];
c, constant;
C_D, drag coefficient, $C_D \equiv 8\tau_w/\rho u^2$;
C_p, heat capacity [kcal/kg °C];
g, gravitational acceleration;
L, m, n, constants;
q, heat flux [kcal/m²h];
t, temperature [°C];
u, velocity in x-direction [m/h];
u_β, velocity head due to buoyancy [m/h]; $u_\beta^2 \equiv \int g\beta(t - t_\infty)\,dx$;
v, velocity in y-direction [m/h];
x, flow direction [m];
y, normal direction to flow [m];
u^*, shear velocity, $u^* \equiv \sqrt{(\tau_w/\rho)}$ [m/h];
u^+, dimensionless velocity, $u^+ \equiv u/u^*$;
t^+, dimensionless temperature, $t^+ \equiv (t_w - t)\,C_p g\tau_w/q_w u^*$;
x^+, dimensionless distance, $x^+ \equiv u^*/v$;
y^+, dimensionless distance, $y^+ \equiv u^*y/v$;
Re_x, Reynolds number, $Re_x \equiv ux/v$;
Nu, Nusselt number, $Nu \equiv \alpha x/\lambda$;
Pr, Prandtl number, $Pr \equiv v/a$;
Gr_x, Grashof number, $Gr_x \equiv gx^3\beta\,(t_w - t_\infty)/v^2$;
Ra_x, Rayleigh number, $Ra_x \equiv Gr_x\,.\,Pr$.

Greek symbols

α, heat-transfer coefficient [kcal/m²h °C];
β, body expansion coefficient [1/°C];
γ, specific weight [kg/m³];
δ, boundary-layer thickness [m];
ε, eddy diffusivity [m²/h];
θ, dimensionless temperature;
λ, thermal conductivity [kcal/mh °C];
v, kinematic viscosity [m²/h];
ρ, density [kg/m³];
τ, shear stress [kg/m²];
δ^+, dimensionless boundary-layer thickness, $\delta^+ \equiv u^*\delta/v$.

Subscripts

cri, critical;
H, energy;
M, momentum;
m, mean;
t, temperature;
u, velocity;
w, wall;

x, local;
∞, infinity.

1. INTRODUCTION

MANY papers about the free convection heat transfer from vertical plate have been presented because it is one of the fundamental heat-transfer problems, but many of them are solved with profile method whose velocity and temperature profiles are given by experiments. For example, Eckert and Jackson [1] solved by assuming the following velocity and temperature profiles:

$$\frac{u}{u_1} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \left(1 - \frac{y}{\delta}\right)^4 \tag{1}$$

$$\frac{t - t_\infty}{t_w - t_\infty} = 1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}. \tag{2}$$

The result is given as

$$Nu_x = 0{\cdot}0295\, Gr_x^{\frac{2}{5}} \,.\, Pr^{\frac{7}{15}} (1 + 0{\cdot}494\, Pr^{\frac{2}{3}})^{-\frac{2}{5}}. \tag{3}$$

As shown by Eckert, the heat-transfer mechanism of free convection has many similar features to those of forced convection. Especially near the wall, we can assume the same turbulent eddy mechanism for free and forced convection because the driving force caused either by pressure gradient or by buoyancy force, has the same effect for the fluid near the wall.

This paper treats the free convection as a special type of forced convection.

2. THEORY

First the following assumptions are made:

(i) The fluid properties are constant except for small change of density.
(ii) The change of velocity and temperature in the flow direction is smaller than that in the normal direction.
(iii) The temperature of the plate is constant.
(iv) The full turbulent region starts from the leading edge of plate.
(v) The eddy diffusivity for momentum ε_M is equal to that for energy ε_H.
(vi) The eddy diffusivity of free convection of a vertical plate is same as that of forced convection of a plate.

Equations for free convection are given as follows:

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

Momentum equation

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial \tau}{\partial y} + \rho g \beta (t - t_\infty), \tag{5}$$

Energy equation

$$C_p \gamma u \frac{\partial t}{\partial x} + C_p \gamma v \frac{\partial t}{\partial y} = \frac{\partial q}{\partial y}. \tag{6}$$

If $\beta = 0$, these equations become the same as those of forced convection. So, the problem of free convection can be solved with the same procedure for forced convection, if the term $\rho g \beta (t - t_\infty)$ is properly treated in the equations. As to forced convection, it is common to solve the following equations instead of solving continuity, momentum and energy equations.

$$\frac{\tau}{\rho} = (\nu + \varepsilon_M) \frac{\mathrm{d}u}{\mathrm{d}y} \tag{7}$$

$$\frac{q}{C_p \gamma} = -(a + \varepsilon_H) \frac{\mathrm{d}t}{\mathrm{d}y} \tag{8}$$

$$\varepsilon_M = \varepsilon_H = \varepsilon = f(y) \tag{9}$$

$$\tau = \tau_w \tag{10}$$

$$q = q_w. \tag{11}$$

By using equations (7–11), velocity and temperature profiles for the y-direction are obtained. Then momentum and energy equations are integrated over the boundary layer, as

$$\frac{\tau_w}{\rho} = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{\delta_t} u(u_\infty - u)\, \mathrm{d}y \tag{12}$$

$$\frac{q_w}{C_p\gamma} = \frac{d}{dx}\int_0^{\delta t} u(t - t_\infty)\, dy. \quad (13)$$

Applying above mentioned method to free convection, the shear stress distribution must be given by equation

$$\frac{\partial \tau}{\partial y} + \rho g \beta (t - t_\infty) = 0 \quad (14)$$

instead of equation (10), as equation (10) is derived by neglecting inertia terms in the momentum equation for forced convection. Other equations, i.e. equations (7–9) and (11), can be applied for free convection without any correction.

Equation (14) will be integrated as

$$\tau = \tau_w - \int_0^y \rho g \beta (t - t_\infty)\, dy \quad (15)$$

because $\tau = \tau_w$ at $y = 0$.
Here, u_β which means accelerated velocity due to buoyancy, is defined as

$$u_\beta \equiv \sqrt{[\int_0^x g\beta(t - t_\infty)\, dx]}. \quad (16)$$

Then its dimensionless term is given as

$$u_\beta^+ = u_\beta / u^* \quad (17)$$

where $u^* = \sqrt{(\tau_w/\rho)}$.

Then equation (15) reduces to

$$\frac{\tau}{\rho} = \frac{\tau_w}{\rho} - \int_0^y \frac{\partial u_\beta^2}{\partial x}\, dy. \quad (18)$$

Changes of shear stress τ_w and heat flux q_w in x-direction are given as

$$\frac{\tau_w}{\rho} = \int_0^{\delta u} g\beta(t - t_\infty)\, dy - \frac{d}{dx}\int_0^{\delta u} u^2 dy \quad (19)$$

$$\frac{q_w}{C_p\gamma} = \frac{d}{dx}\int_0^{\delta t} u(t - t_\infty)\, dy. \quad (20)$$

Using shear velocity u^*, equations (7, 8, 18, 19, 20) are converted to dimensionless forms as

$$\frac{\tau}{\tau_w} = \left(1 + \frac{\varepsilon}{\nu}\right)\frac{du^+}{dy^+} \quad (21)$$

$$\frac{q}{q_w} = 1 = \left(\frac{1}{Pr} + \frac{\varepsilon}{\nu}\right)\frac{dt^+}{dy^+} \quad (22)$$

$$\frac{\tau}{\tau_w} = 1 - \int_0^{y+} \frac{\partial u_\beta^{+2}}{\partial x^+}\, dy^+ \quad (23)$$

$$x^+ = \int_0^{\delta u+} (u_\beta^{+2} - u^{+2})\, dy^+$$

$$= \int_0^{\delta t+} u^+(t_\infty^+ - t^+)\, dy^+. \quad (24)$$

The eddy diffusivity distribution which can be applied to either forced or free convection is given by the following equation.

$$\varepsilon/\nu = 0{\cdot}4\, y^+[1 - \exp(-0{\cdot}0017\, y^{+2})]. \quad (25)$$

ε/ν in equation (25) is proportional to $0{\cdot}4\, y^+$ at the distance of the wall, and to $0{\cdot}00068\, y^{+3}$ at the vicinity of the wall. Equation (23), as a first approximation, can be reduced as follows,

$$\frac{\tau}{\tau_w} = 1 - \int_0^{y+} \frac{u_\beta^{+2}}{x^+}\, dy^+ \quad (26)$$

which assumes u_β^{+2} is proportional to x^+.

Dimensionless velocity u^+, dimensionless temperature t^+, Grashof number Gr_x, and Nusselt number Nu_x are given as follows.

$$u^+ = \int_0^{y+} \frac{1 - \int_0^{y+} \frac{u_{\beta w}^{+2}}{x^+}\left(1 - \frac{t^+}{t_\infty^+}\right) dy^+}{1 + 0{\cdot}4\, y^+[1 - \exp(-0{\cdot}0017\, y^{+2})]} \times dy^+ \quad (27)$$

$$t^+ = \int_0^{y+} \frac{dy^+}{1/Pr + 0{\cdot}4\, y^+[1 - \exp(-0{\cdot}0017\, y^{+2})]} \quad (28)$$

$$Gr_x = \frac{x^3 g\beta(t_w - t_\infty)}{\nu^2} = (u_{\beta w}^+ \cdot x^+)^2 \quad (29)$$

$$Nu_x = \frac{\alpha_x x}{\lambda} = \frac{x^+ Pr}{t_\infty^+}. \quad (30)$$

Equations (24, 27–30) are solved by the trial and error method. First the value of $u_{\beta w}^{+2}/x^+$ is assumed and velocity and temperature profiles are calculated. Then the value of x^+ evaluated with velocity is compared with that evaluated with temperature. If they are not similar, a new corrected value of $u_{\beta w}^{+2+}/x^+$ is assumed. Computations are done with HITAC7020E digital computer at the University of Tokyo.

which is divided by a certain distance where velocity becomes half of the maximum velocity. The ordinate is dimensionless velocity or dimensionless temperature which are divided by their maximum values respectively. Experimental results are referred from the paper by Eckert [1], so, the exact Prandtl number of experiments cannot be known. Here, they are compared with theoretical ones whose Prandtl number is unity. The agreement of velocity profile is good, but that for temperature is poor.

Figure 3 shows the comparison between the

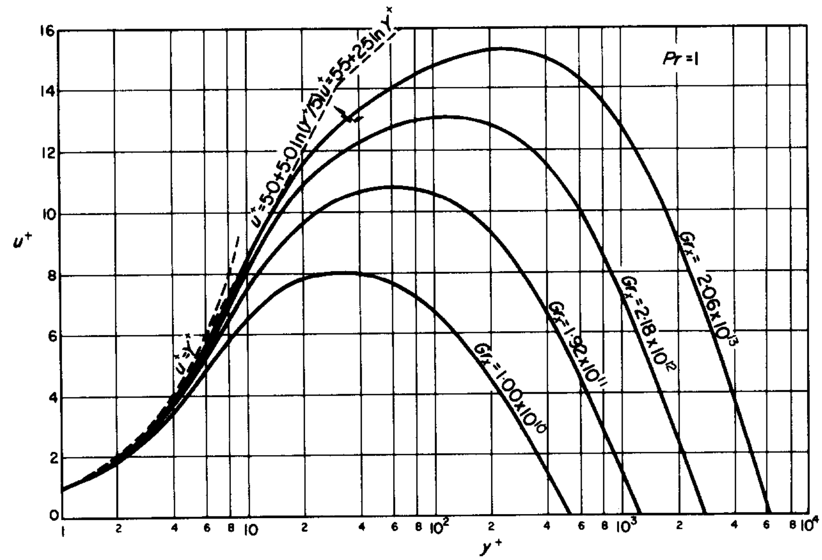

FIG. 1. Theoretical velocity profile.

3. RESULTS

Figure 1 shows dimensionless velocity profile u^+ by equation (27) for various Grashof numbers where the Prandtl number remains unity. Near the wall, the velocity profile becomes asymptotic to the straight line, $u^+ = y^+$. For the larger Grashof numbers, the velocity profile at a short distance from the wall takes the form of pattern as that of forced convection.

Figure 2 shows the experimental velocity and temperature profile by Griffith. The abscissa is the dimensionless distance from the wall, theoretical temperature profile and experimental ones of water which were made by Fujii [2] very recently. The abscissa is dimensionless distance from the wall whose definition is given in Fig. 3. The agreement between theory and experiment is good.

Figure 4 shows the theoretical local heat-transfer coefficient given by equation (30) for various Grashof and Prandtl numbers. For lower Grashof and Prandtl number region the above mentioned equation is no longer true, because the flow remains laminar at the region.

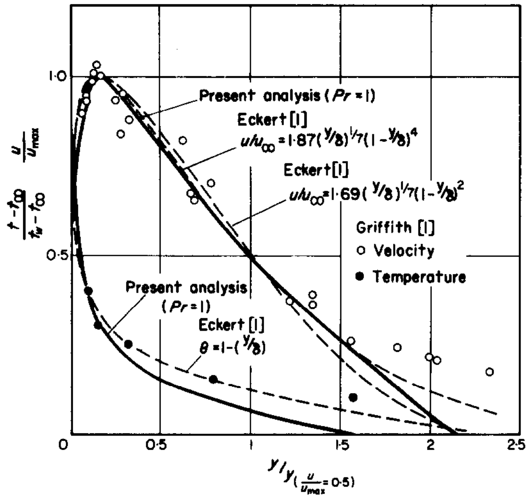

FIG. 2. Comparison between experiments and theoretical velocity and temperature profiles.

The chain line shows the proposed transition line from laminar to turbulent. More detailed discussion on the transition will be given in the following section. The broken lines in the figure refer to the approximate equation, given later. For large Prandtl number, the Nusselt number is proportional to the 0·215 power of Prandtl number ($Nu \propto Pr^{0.215}$), which is a smaller power than the result by Eckert. On the other hand, for very small Prandtl numbers, the power becomes 0·555, which is a little larger than that by Eckert.

Here, a simple approximate equation is given in the form

$$Nu_x = cGr_x^m(Pr^n - L) \tag{31}$$

where c, L, m and n are constants which are shown in Table 1.

By assuming that the turbulent region starts from the leading edge, the mean Nusselt number is given as

$$Nu_m = c'\,Gr_x^m(Pr^n - L) \tag{32}$$

where the constant c' is also given in Table 1.

Figure 5 shows the comparison between equation (32) and experimental results by various authors [3–6]. Figure 5(a) and (b) show the mean Nusselt number for Prandtl number nearly one and nearly 40 respectively. Good agreements are obtained for both cases. Figure

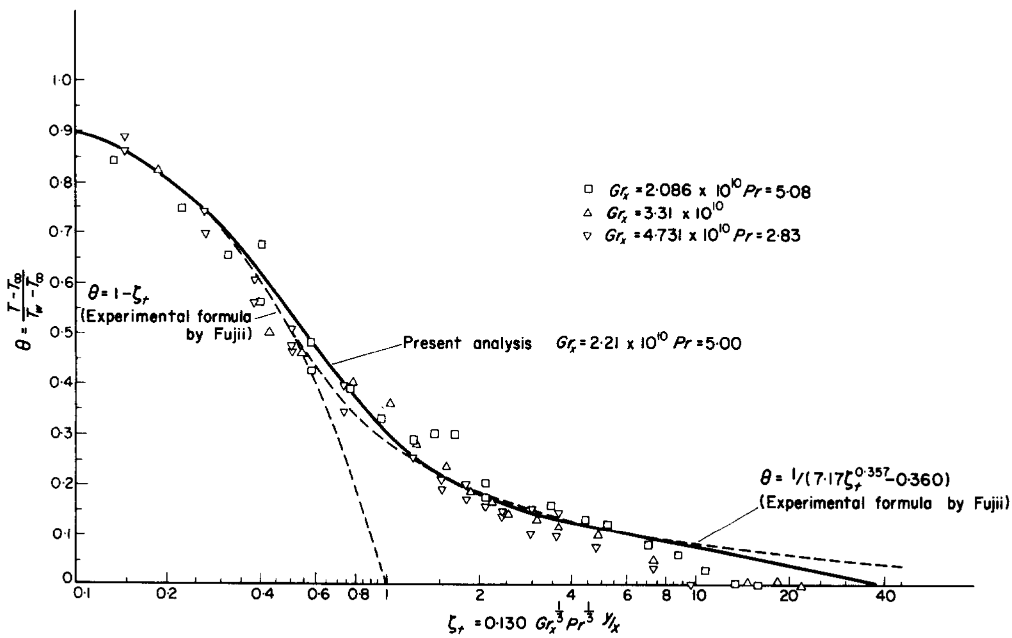

FIG. 3. Temperature profile with Fujii's experiments.

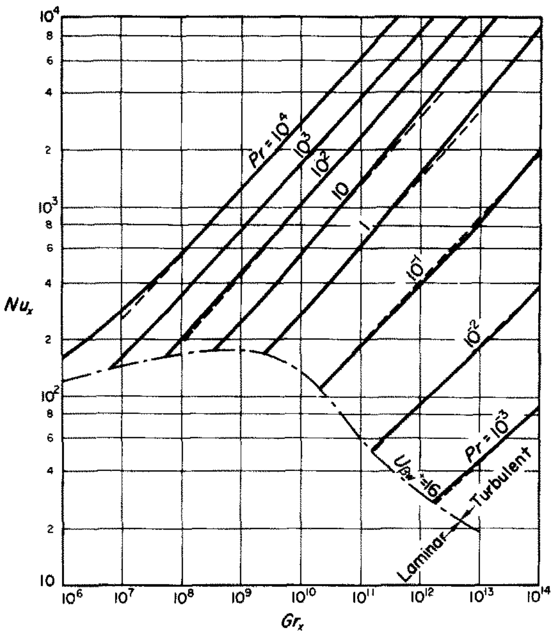

FIG. 4. Local heat-transfer coefficient.

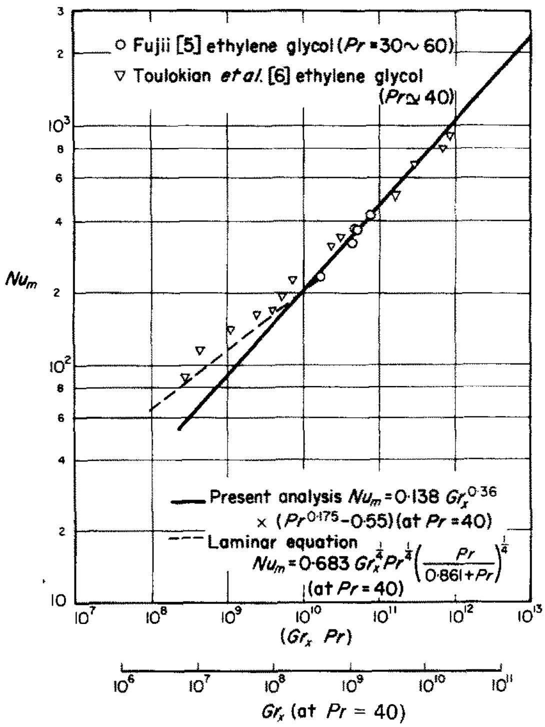

FIG. 5(b). Comparison with experiments ($Pr \simeq 40$)

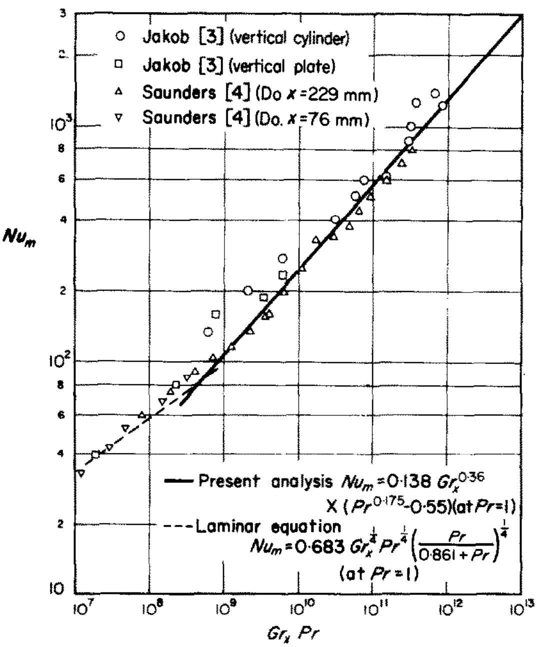

FIG. 5(a). Comparison with experiments ($Pr \simeq 1$)

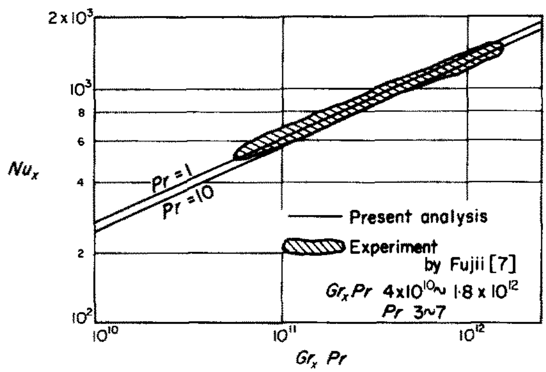

FIG. 5(c). Comparison with experiments (Pr = 3 ~ 7).

Table 1. Coefficients in equations (31) *and* (32)

$(10^{10} < Ra_x < 10^{13})$

Pr	c	m	n	L	c'
$\sim 3 \times 10^{-3}$	0·185	0·31	0·55	0	0·199
$3 \times 10^{-3} \sim 3 \times 10^{-2}$	0·176	0·32	0·55	0	0·183
$3 \times 10^{-2} \sim 3 \times 10^{-1}$	0·114	0·34	0·55	0	0·112
$3 \times 10^{-1} \sim 10^2$	0·149	0·36	0·175	0·55	0·138
$10^2 \sim 10^4$	0·30	0·31	0·215	0	0·323

5(c) is the result for water by Fujii *et al.* [7], where the local Nusselt number is compared with equation (31).

4. TRANSITION FROM LAMINAR TO TURBULENT

Here, the discussion about transition from laminar to turbulent will be done. Many experiments show that the transition occurs when the Rayleigh number of fluid reaches a certain value, say 10^{10}. Hermann [8] showed that the transition occurs at a certain Reynolds number which is defined by maximum velocity and thickness of boundary layer. Fujii [9] says that the eddy occurs in the laminar boundary layer first and it causes the transition to turbulent.

It is common that Reynolds number is used as a criterion for transition in forced convection. For example, the critical Reynolds number of pipe flow is about $2{\cdot}3 \times 10^3 \sim 10^4$.

Considering the laminar, transitional and the turbulent flow region the increase of Reynolds number with increase of velocity is linear. Though the decrease of friction coefficient of C_f is almost linear in the laminar region, C_f becomes discrete in the transitional zone for the same Reynolds number. And again in the turbulent zone C_f decreases gradually with increase of velocity.

Then, the friction coefficient can also be used as a criterion of transition. Here, as a shear velocity u_b^+ is given by

$$u_b^+ = \frac{u_b}{u^*} = \sqrt{\left(\frac{\rho u_b^2}{\tau_w}\right)} = \sqrt{\left(\frac{8}{C_D}\right)} \qquad (33)$$

where C_D is coefficient, the drag shear velocity can also be another criterion for transition. For example lower critical Reynolds number of pipe flow is 2300, which corresponds to shear velocity $u_b^+ = 13{\cdot}2$. The shear velocity shows the rate between mean flow energy and dissipated energy at wall, as seen in the expression of equation (33).

If one takes same criterion for free convection, $u_{\beta w}^+$ must be chosen as

$$u_{\beta w}^+ = \frac{u_{\beta w}}{u^*} = \sqrt{\left(\frac{\rho u_{\beta w}^2}{\tau_w}\right)}$$

$$= \sqrt{\left[\frac{\int_0^x \rho g \beta (t_w - t_\infty)\,\mathrm{d}x}{\tau_w}\right]} \qquad (34)$$

It shows the rate between potential energy due to buoyancy and dissipated energy at wall.

From present theory $u_{\beta w}^+$ can be calculated for various Grashof and Prandtl number as shown in Fig. 6.

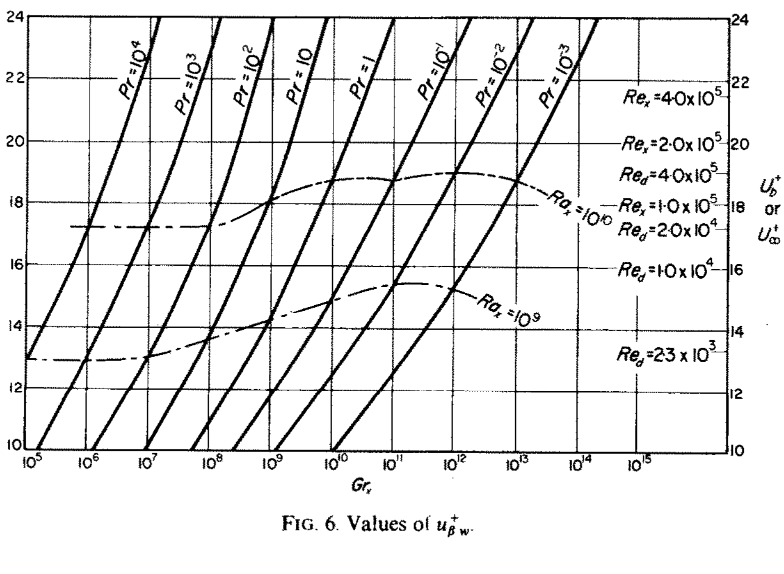

FIG. 6. Values of $u_{\beta w}^+$.

Chain lines in Fig. 6 show iso-Rayleigh number curves. The figures shown in the right hand side are shear velocities corresponding to Reynolds number for forced convection, where Re_x and Re_d are for plate and in-pipe flow respectively.

Fujii showed that the Rayleigh number for laminar to eddy turbulent was 10^9 by his experiment [9]. It is interesting that the value of corresponding $u_{\beta w}^+$ is almost same that of u_b^+ for $Re_d = 2300$, the lower critical Reynolds number of pipe flow.

Figure 7 represents the experimental critical Grashof number to Prandtl number. Continuous lines are iso-$u_{\beta w}^+$ curves, and broken lines are iso-Rayleigh number curves. From experimental results, the line $u_{\beta w}^+ = 16{\cdot}0$ is chosen as the transition line. Critical Rayleigh number becomes larger Prandtl number as shown in Fig. 7.

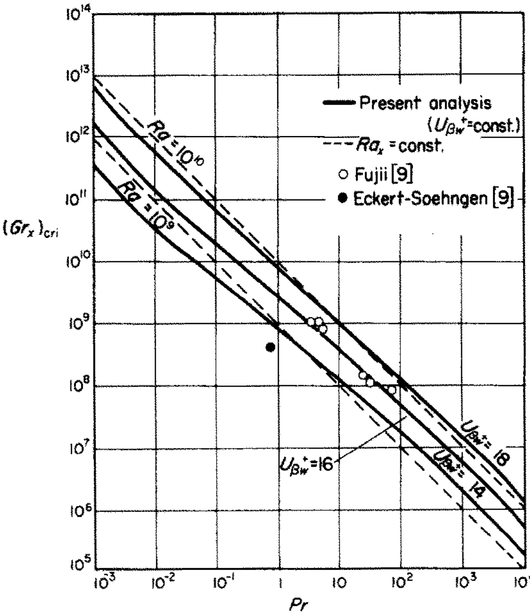

FIG. 7. Values of $u_{\beta w}^+$ at transition from laminar to turbulent.

5. CONCLUSION

The heat-transfer coefficient of free convection from a vertical plate is evaluated theoretically without using the profile method. The shear stress, heat flux, and eddy diffusivity distribution are assumed using the result of forced convection heat transfer. The obtained velocity and temperature profile and heat-transfer coefficient agree well with experimental results given by various authors.

On transition from laminar to turbulent, a new criterion is proposed which gives the ratio of energy of mean flow and that dissipated near the wall. It can be applied for both forced and free convection.

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Résumé—Jusqu'à présent, le problème du transport de chaleur turbulent en convection naturelle a été résolu par la méthode des profils à l'aide des profils de vitesse et de température donnés par l'expérience. Dans cet article, la contrainte de cisaillement, le flux de chaleur et la distribution de diffusivité turbulente sont pris en utilisant le résultat du transport de chaleur par convection forcée. Les équations aux dérivées partielles couplées pour la vitesse et la température qui sont ramenées à des formes plus simples, peuvent être facilement résolues par la méthode des essais et corrections successifs. Le résultat peut être appliqué pour une gamme de nombres de Grashof et de Prandtl plus large que celle obtenue à partir des équations habituelles.

Pour la transition du laminaire au turbulent, on propose un certain critère sans dimensions, qui donne le rapport de l'énergie de l'écoulement moyen à celle dissipée près de la paroi.

Le même nombre sans dimensions peut être appliqué à la fois à la convection forcée et à la convection naturelle.

Zusammenfassung—Bis heute wurde das Problem des turbulenten Wärmeübergangs durch freie Konvektion mit Hilfe der Profilmethode gelöst, wobei die Geschwindigkeits- und Temperaturprofile aus Versuchen erhalten wurden. In dieser Arbeit werden Schubspannung, Wärmestromdichte und turbulenter Austauschgrad auf Grund der Ergebnisse für den Wärmeübergang bei Zwangskonvektion angenommen, worauf die gekoppelten partiellen Differentialgleichungen für die Geschwindigkeit und die Temperatur, die auf einfachere Formen reduziert wurden, leicht durch Probieren gelöst werden können. Die Ergebnisse können auf einen grösseren Bereich von Grashof- und Prandtl-Zahlen angewandt werden, als jene die nach den herkömmlichen Gleichungen erhalten wurden.

Für den Umschlag von laminar nach turbulent wird ein dimensionsloses Kriterium vorgeschlagen, welches das Verhalten der Energie der Mittelströmung zu jener nahe der Wand angibt. Der gleiche dimensionslose Wert kann für erzwungene und freie Konvektion verwendet werden.

Аннотация—До настоящего времени задача о турбулентном теплообмене ири псвободной конвекции решалась интегральными методами, а профили скорости и температуры определялись экспериментально. В данной статье принимаются распределения касательного напряжеия, теплового потока и турбулентной диффузии в соответствии с результатами по теплобмену при вынужденной конвекции. При этом методом проб и ошибок можно легко решить систему дифференциальных уравнений в частных производных для скорости и температуры, записанных в более простой форме. Этот подход можно применить для более широкого диапазона изменения чисел Грасгофа и Прандтля, чем результат решения обычных уравнений.

При переходе от ламинарного режима к турбулентному вводится безразмерный критерий, который представляет отношение энергии основного потока к энергии вблизи стенки. Эту же безразмерную величину можно применить как для вынужденной, так и свободной конвекции.